Q1 (a) $\frac{a_0}{2} + \sum_{n=1}^{\infty}(a_n \cos nx + b_n \sin nx)$

$a_n = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\cos nx\,dx$, $b_n = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\sin nx\,dx$

(b) sign(x) is odd so $a_n = 0$ for all n.

$b_n = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\sin nx\,dx = \frac{2}{\pi}\int_0^{\pi}\sin nx\,dx = \frac{2}{\pi}\left[\frac{-\cos nx}{n}\right]_0^{\pi}$

$= \frac{2}{\pi}\frac{1}{n}(1-(-1)^n) = \begin{cases} \frac{4}{n\pi} & n \text{ odd} \\ 0 & n \text{ even} \end{cases}$

$f(x) = \frac{4}{\pi}\sum_{k=1}^{\infty}\frac{\sin[(2k-1)x]}{2k-1}$

(c) PI states $\frac{1}{\pi}\int_{-\pi}^{\pi} f^2(x)\,dx = \frac{a_0^2}{2} + \sum_{n=1}^{\infty}(a_n^2 + b_n^2)$

Proof: $\frac{1}{\pi}\int_{-\pi}^{\pi} f^2(x)\,dx = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\left(\frac{a_0}{2} + \sum_{n=1}^{\infty}(a_n\cos nx + b_n\sin nx)\right)dx$

$= \frac{a_0^2}{2} + \sum_{n=1}^{\infty}\frac{1}{\pi}\int_{-\pi}^{\pi} f(x)(a_n\cos nx + b_n\sin nx)\,dx$

$= \frac{a_0^2}{2} + \sum_{n=1}^{\infty}(a_n^2 + b_n^2)$ q.e.d.

(d) $\frac{1}{\pi}\int_{-\pi}^{\pi}\text{sign}^2 x\,dx = \frac{1}{\pi}2\pi = 2 \underset{(P.I.)}{=} 0 + \sum_{k=0}^{\infty}\left(0 + \left(\frac{4}{\pi(2k+1)}\right)^2\right)$

$= \frac{16}{\pi^2}\sum_{k=0}^{\infty}\frac{1}{(2k+1)^2} \Rightarrow \frac{\pi^2}{8} = \sum_{k=0}^{\infty}\frac{1}{(2k+1)^2}$ q.e.d.

Q2 (a) $\varepsilon_{ijk}\varepsilon_{klm} = \delta_{il}\delta_{jm} - \delta_{im}\delta_{jl}$

(b) $(\text{grad}(\underline{A}\cdot\underline{B}))_i = \partial_i (A_j B_j) = (\partial_i A_j) B_j + A_j \partial_i B_j$

$(\underline{B}\cdot\nabla\underline{A} + \underline{A}\cdot\nabla\underline{B} + \underline{B}\times \text{curl}\,\underline{A} + \underline{A}\times\text{curl}\,\underline{B})_i$

$= B_j \partial_j A_i + A_j \partial_j B_i + \varepsilon_{ijk} B_j (\text{curl}\,\underline{A})_k + \varepsilon_{ijk} A_j (\text{curl}\,B_k)$

$= \quad " \quad + \quad " \quad + \varepsilon_{ijk}\varepsilon_{klm} B_j \partial_l A_m + \varepsilon_{ijk}\varepsilon_{klm} A_j \partial_l B_m$

$= \quad " + " + (\delta_{il}\delta_{jm} - \delta_{im}\delta_{jl})(B_j \partial_l A_m + A_j \partial_l B_m)$

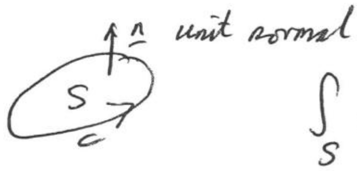
$= \underline{B_j \partial_j A_i} + \underline{A_j \partial_j B_i} + B_j \partial_i A_j - \underline{B_j \partial_j A_i}$

$+ A_j \partial_i B_j - \underline{A_j \partial_j B_i}$

$= A_j \partial_i B_j + B_j \partial_i A_j = (\text{grad}\,\underline{A}\cdot\underline{B})_i$

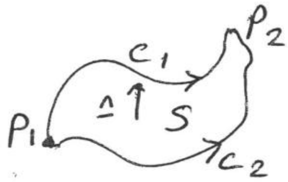
q.e.d

Q3 (a)



$$\int_S \operatorname{curl}\underline{A}\cdot\underline{n}\,dS = \oint_c \underline{A}\cdot d\underline{r}$$

(b) $\displaystyle\int_c \operatorname{grad}\varphi\cdot d\underline{r}$



$$= \int_S \operatorname{curl}(\operatorname{grad}\varphi)\cdot\underline{n}\,dS$$

where S is the surface between c_1 and c_2 and $c = c_2 + (-c_1)$ where "$-c_1$" is c_1 in reverse.

The S-integral is zero as $\operatorname{curl}\operatorname{grad}\varphi = 0$. Thus

$$0 = \int_c \operatorname{grad}\varphi\cdot d\underline{r} = \int_{c_2}\operatorname{grad}\varphi\cdot d\underline{r} - \int_{c_1}\operatorname{grad}\varphi\cdot d\underline{r}$$

So $\displaystyle\int_{c_1}\operatorname{grad}\varphi\cdot d\underline{r} = \int_{c_2}\operatorname{grad}\varphi\cdot d\underline{r}$. q.e.d.

c) $$\frac{\underline{r}}{|\underline{r}|^3} + x\underline{i} = \nabla\Big(\underbrace{\tfrac{1}{2}x^2 - \frac{1}{|\underline{r}|}}_{\varphi\text{ say}}\Big)$$

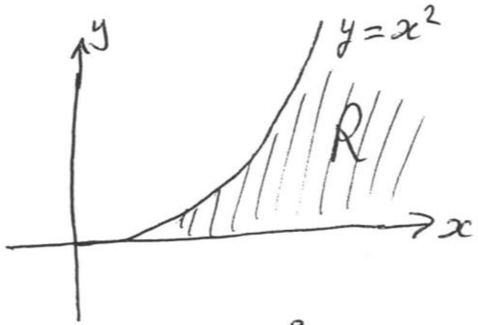
$$\int_C \nabla\varphi\cdot d\underline{r} = \varphi(1,2,3) - \varphi(0,1,2)$$

$$= \frac{1}{2} - \frac{1}{\sqrt{1^2+2^2+3^2}} - 0 + \frac{1}{\sqrt{0^2+1^2+2^2}}$$

$$= \frac{1}{2} - \frac{1}{\sqrt{14}} + \frac{1}{\sqrt{5}}$$

Q4(a) $\dfrac{\partial(u,v)}{\partial(x,y)} = \begin{vmatrix} \frac{\partial u}{\partial x} & \frac{\partial v}{\partial x} \\ \frac{\partial u}{\partial y} & \frac{\partial v}{\partial y} \end{vmatrix}$

(b)



$x = \sqrt{y} \Rightarrow y = x^2$

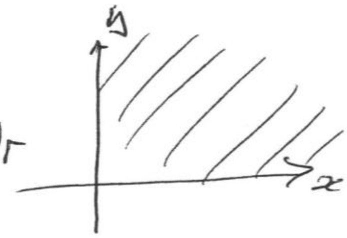
$$\int_0^\infty \int_{\sqrt{y}}^\infty e^{-\mu x^3} \, dx\, dy = \int_0^\infty \int_0^{x^2} e^{-\mu x^3} \, dy\, dx$$

$$= \int_0^\infty x^2 e^{-\mu x^3} \, dx = \frac{1}{3} \int_0^\infty e^{-\mu x^3} \, d(x^3)$$

$$= \frac{1}{3\mu}.$$

(c) Use polars $x = r\cos\theta$, $y = r\sin\theta$

$\left| \dfrac{\partial(x,y)}{\partial(r,\theta)} \right| = r$



$$\int_0^\infty \int_0^\infty e^{-r^2} \, dx\, dy = \int_0^\infty \int_0^{\pi/2} e^{-r^2} r \, d\theta \, dr$$

$$= \int_0^\infty \frac{\pi}{2} e^{-r^2} r \, dr$$

$$= \frac{\pi}{2} \int_0^\infty e^{-r^2} \frac{dr^2}{2} = \frac{\pi}{4}$$

Q5 (a)

$\underline{n}$ unit outward normal

$$\int_V \operatorname{div} \underline{A}\, dV = \int_S \underline{A}\cdot\underline{n}\, ds$$

(b) Let $x = 1+X,\ y = 1+Y,\ z = 1+Z$

In (X, Y, Z) V is the unit ball.

$$\underline{A}(X,Y,Z) = (1+X+1+Y)\underline{i} + ((1+X)^2+(1+X)(1+Y))\underline{j} + (1+Z)^2\underline{k}$$

$$= (2+X+Y)\underline{i} + (1+2X+X^2+1+X+Y+XY)\underline{j} + (1+2Z+Z^2)\underline{k}$$

$$= (2+X+Y)\underline{i} + (2+3X+X^2+Y+XY)\underline{j} + (1+2Z+Z^2)\underline{k}$$

$$\operatorname{div} \underline{A} = 1 + 1+X + 2+2Z$$

So $\displaystyle\int_V \operatorname{div}\underline{A}\, dV = \int_V (4+X+2Z)\, dV = 4|V| = 4\,\frac{4\pi}{3}$

$= \dfrac{16\pi}{3}$ (by symmetry). $\displaystyle\int_S \underline{A}\cdot\underline{n}\, ds = \int_S \underline{A}\cdot\hat{\underline{R}}\, dS$

$$= \int_S ((2+X+Y)X + (2+3X+X^2+Y+XY)Y + (1+2Z+Z^2)Z)\, ds$$

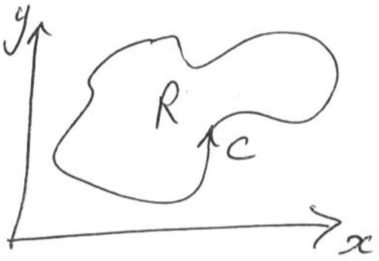
As $S \to S$ under $X\to -X$, or $Y \to -Y$, or $Z \to -Z$,

$\displaystyle\int_S = \int_S (X^2+Y^2+2Z^2)\, dS$ Also $X^2+Y^2+Z^2=1$ and $\displaystyle\int_S X^2 dS = \int_S Y^2 dS = \int_S Z^2 dS$

$$= \left(\frac{1}{3}+\frac{1}{3}+\frac{2}{3}\right)|S| = \frac{4}{3}|S| = \frac{4\,4\pi}{3} = \frac{16\pi}{3}$$

$= \int_V$ qed.

Q6(a)



$$\oint_C (P\,dx + Q\,dy) = \iint_R \left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y}\right) dx\,dy$$

(b) Use Stokes theorem with $\underline{n} = \underline{k}$, $S = R$

and $\underline{A} = (P, Q, 0)$.

$$\int_S \underline{A}\cdot d\underline{r} = \int_C (P\,dx + Q\,dy). \quad \text{curl}\,\underline{A} = \begin{vmatrix} \underline{i} & \underline{j} & \underline{k} \\ \partial_x & \partial_y & 0 \\ P & Q & 0 \end{vmatrix}$$

$$\Rightarrow \underline{k}\cdot \text{curl}\,\underline{A} = \begin{vmatrix} \partial_x & \partial_y \\ P & Q \end{vmatrix} = \partial_x Q - \partial_y P$$

So $\displaystyle\int_S \underline{k}\cdot\text{curl}\underline{A}\,dS = \int_R \left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y}\right) dx\,dy$

$\displaystyle = \oint_C (P\,dx + Q\,dy)$ by Stokes' theorem.

(c) $\displaystyle\oint_C (y\,dx + x(2+y)\,dy) = \int_0^{2\pi} (\sin\theta\, d\cos\theta + \cos\theta\,(2+\sin\theta)\, d\sin\theta)$

$\displaystyle = \int_0^{2\pi} (-\sin^2\theta + 2\cos^2\theta + \cos\theta\sin\theta\cos\theta)\, d\theta$

$\displaystyle = 2\pi\left[-\frac{1}{2} + 2\,\frac{1}{2} + 0\right] = \pi$

$\displaystyle\iint_R = \int_R [(2+y) - 1]\, dx\,dy = \int_R (1+y)\, dx\,dy = 1\,|R| = \pi$

$\displaystyle = \oint_C$ q.e.d.